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Numerical Solution of Periodically Pulsed Laminar Free Jets

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ALTHOUGH the behavior of a steady, two-dimensional, laminar free jet has been treated in detail by many workers such as Pai[1] and Schlichting,[2] there is still limited information on the flow development region near the nozzle exit, the effect of exit conditions on flow characteristics, the location of the virtual origin, and the location at which the velocity profile becomes very nearly self-preserving. Few detailed experimental studies have been reported, mainly because the laminar free jet has limited applications. Numerical solutions have been sought by Pai and Hsieh[3] and Hornbeck.[4]

Velocity profiles fluctuating about a mean are common in practice. However, unsteady, two-dimensional, laminar free jets issuing into a stationary medium have scarcely been explored either experimentally or theoretically. This is mainly because unsteady laminar jets normally become turbulent shortly after leaving the nozzle. Closed form solutions always entail certain assumptions which limit their range of validity. For instance, McCormack et al.[5] obtained a closed solution for the mean velocity of a gas jet in the near-field region with a source of periodicity at the orifice by employing the method developed by Lin.[6] However, the solution is only valid for high frequencies. The problem of unsteady mixing of compressible fluid has been studied by Pai[7] with perturbation techniques. Recently, Kent[8] computed the unsteady laminar axisymmetric jet by using an integral method which reduces the number of spatial coordinates from two to one. This greatly reduces computation in terms of space and time but suffers from the drawback that accurate velocity profile information cannot be obtained.

The objectives of this study are to use the case of a laminar free jet to develop an efficient transformation technique which is applicable to the computation of unsteady turbulent jets, to study the mean flow development of steady and unsteady laminar free jets, and to examine the unsteady effects.

Governing Equations

Consider a two-dimensional, constant property laminar free jet issuing into a stationary medium with a velocity field which is perturbed from an initially steady value $u_i(x,y)$ by a sinusoidal mass flow variation of amplitude ϵ superimposed at the nozzle exit. Following Schlichting,[2] an order of magnitude analysis indicates that to a first approximation, the streamwise pressure gradient term can be neglected. Hence, the pressure ceases to be an unknown function and the nondimensional thin shear layer equations are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Here u and v are instantaneous velocities in the streamwise x and transverse y direction and are nondimensionalized with respect to the mean centerline velocity at the nozzle, $\bar{u}^*_{ci}$. All length scales are nondimensionalized by the nozzle half-width, h_w. The nondimensional time t and circular frequency ω of the nozzle mass flow perturbation are referred to h_w^2/ν and ν/h_w^2, respectively, where ν is the kinematic viscosity.

The boundary conditions on the centerline and at the edge of the jet can be expressed by

$$t\geq 0 \begin{cases} y=0,\ \partial u/\partial y=v=0 & \text{(3a)}\\ y=\infty,\ u=0 & \text{(3b)}\end{cases}$$

The time-varying boundary condition at the nozzle exit is given by

$$t\geq 0,\ x=0,\ u=\bar{u}_0(y)(1+\epsilon\sin\omega t) \tag{4a}$$

Here $\bar{u}_0(y)$ is the nondimensional mean velocity profile at the nozzle exit.

For all locations downstream of the nozzle, the initial conditions are given by

$$t=0,\ x\geq 0,\ u=u_i(x,y) \tag{4b}$$

Transformed Variable Form

Because the jet spreads away from its centerline, it is advantageous to transform Eqs. (1-4) so that the width of the jet in the transformed spatial coordinates (ζ,η) is almost independent of the streamwise distance ζ. Thus, the velocity profile at any downstream ζ-station can be represented by velocities at a set of predetermined transverse locations. The transformation variables used are based on the similarity solution for the steady jet by Bickley[9] because the spreading rate of the jet is known where the similarity solution is reached. Hence, a dimensionless transverse distance η and a dimensionless stream function f are defined by

$$\eta=ay(\zeta+\zeta_0)^{-2/3} \tag{5}$$

$$\psi(\zeta,t,y)=b(\zeta+\zeta_0)^{1/3}f(\zeta,t,\eta) \tag{6}$$

where a, b, and ζ_0 are arbitrary constants which can be varied to facilitate computation, x has been renamed as ζ, and ψ is the dimensionless stream function in the (ζ,t,y) coordinates.

The function f automatically satisfies Eq. (1). By choosing $a/b=1/3$ for convenience, Eq. (2) can be rewritten as

$$f'''+(f')^2+ff''=3(\zeta+\zeta_0)\times\left[f'\frac{\partial f'}{\partial\zeta}-f''\frac{\partial f}{\partial\zeta}+\frac{(\zeta+\zeta_0)^{1/3}}{ab}\frac{\partial f'}{\partial t}\right] \tag{7}$$

where prime denotes differentiation with respect to η.

The boundary conditions in Eq. (3) become

$$t\geq 0 \begin{cases} \eta=0,\ f''=0,\ f+3(\zeta+\zeta_0)\dfrac{\partial f}{\partial\zeta}=0 & \text{(8a)}\\ \eta=\infty,\ f'=0 & \text{(8b)}\end{cases}$$

The boundary conditions at $\zeta=0$ in the (η,t) plane are obtained by writing Eq. (4a) as

$$f'=\bar{f}'_0(\eta)(1+\epsilon\sin\omega t) \tag{9}$$

The initial conditions at $t=0$ in the (ζ,η) plane [Eq. (4b)] are generated by the solutions $f'_i(\zeta,\eta)$ of the following steady jet

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equation obtained from Eq. (7) by eliminating the time-dependent term.

$$f''' + (f')^2 + ff'' = 3(\zeta + \zeta_0)\left[f' \frac{\partial f'}{\partial \zeta} - f'' \frac{\partial f}{\partial \zeta}\right] \quad (10)$$

Method of Solution

Because the initial conditions along the (ζ,η) plane are generated by solving the steady jet Eq. (10) subject to Eqs. (8) and (9), the streamwise flow development of the steady jet can also be studied. Both Eqs. (7) and (10) are parabolic and can be solved by a marching procedure. The finite difference scheme employed is the Box Method developed by Keller,[10] and described in detail by Cebeci and Bradshaw.[11] The scheme has been applied successfully to two-dimensional time-dependent flows by Cebeci.[12]

Results and Discussion

Steady Jet

Two initial velocity profiles $\bar{u}_0(y)$ are investigated, namely parabolic and rectangular profiles. Convergence of the velocity profile is satisfied at each ζ-station after two or three iterations except for the initial development where five or six iterations are required. In both cases, the sensitivity of the solution to different grid sizes was tested and the momentum in the streamwise direction was found to be conserved to within about 1%. For the initially parabolic profile, only 24 grid points in the transverse η-direction were used throughout the computation. In terms of storage and computing time, the present method, through the employment of a transformation and the Keller Box numerical scheme, shows substantial improvement over the method of Pai and Hsieh.[3] They started their solution at $\zeta = 0$ with 200 grid points in the transverse η-direction and as the computation proceeded downstream, this number was increased to 4000.

The computed decay of centerline velocity u_c in Fig. 1 is in excellent agreement with that of Pai and Hsieh.[3] Very little experimental data have been reported. One set of results from Chanaud and Powell,[13] corresponding to their stated Reynolds number of 68 based on average nozzle velocity, agrees fairly well with the numerical solution (Fig. 1) after a correction. From the reported exit velocity profile of Chanaud and Powell,[13] the Reynolds number based on the centerline velocity should be 114.

A plot of computed values of u_c^{-3} vs ζ is essentially a straight line for $0.07 \leq \zeta \leq 10.1$ and extrapolation indicates that the virtual origin is at $\zeta_v = -0.088$. Hence, the starting point of self-preservation is given by $\zeta_s \simeq 0.07$. Unfortunately, experimental data for the location of the virtual origin and the starting point of self-preservation for the laminar free jet are not available. The similarity solution plotted in Fig. 1 using the above location of the virtual origin indicates that self-preservation is actually attained earlier than would have been predicted by assuming the virtual origin to be at the nozzle exit.

For the initially rectangular profile, the maximum number of grid points used in the transverse direction was only 38. The self-preserving velocity profile is obtained at $\zeta_s \simeq 0.3$ and the virtual origin is located at $\zeta_v = -0.222$. As shown in Fig. 1, the numerical results agree well with those of Pai and Hsieh.[3]

Unsteady Jet

Only a parabolic mean velocity profile $\bar{u}_0(y)$ at the nozzle with sinusoidal perturbation of frequency ω is considered.[14] The steady-state solution is regarded as attained when the velocity profiles repeat for a few periods. The maximum number of grid points used in the transverse direction is 30 and the convergence rate is comparable to that of the steady jet. In the absence of guidance from available experimental and numerical data, results were obtained for $\omega = 0.1$, 1, 5, 10, and 100 with $\epsilon = 0.1$. In all the computed cases at various frequencies, the mean momentum flux in the streamwise direction is conserved to within 1%.

At all the computed frequencies, the mean velocity profiles, mean entrainment, mean centerline velocity, and mean jet spreading for the periodically pulsed jet are found to differ insignificantly from those for the steady jet. A set of calculations for $\omega = 10$ with ϵ increased to 0.2 again yielded mean results which follow closely those for the steady jet. This is perhaps due to the absence of large scale vortices induced within the range of frequencies and amplitude of pulsation tested. However, for jets oscillating with high frequencies ($\omega > 5$) or at large streamwise distance ($\zeta > 1$) from the nozzle, instantaneous velocity profiles depart significantly from quasisteady values. Here quasisteady solutions over a region of interest refer to values approximated by a sequence of steady jet solutions, each of which corresponds to the instantaneous conditions at the nozzle. The phase angle between the steady-state fundamental component of the centerline velocity at any streamwise station and that at the nozzle exit is a lag which increases with both frequency and streamwise distance.

Unsteady effects are also apparent in the form of harmonics which distort the waveform for the variation of the steady-state instantaneous centerline veclocity with time. The peak-to-peak oscillation, F, of the steady-state instantaneous centerline velocity expressed as a percentage of the mean value at any downstream station is much larger than its initial value of 20% at the nozzle. A plot of the variation of F with ζ is shown in Fig. 2. Although the results are scattered, they do show a trend. For a given ζ, F increases with frequency. Furthermore, F increases with ζ and for large ω it appears to increase at a slower rate if ζ is sufficiently large, say $\zeta > 1$. However, for lower frequency such as $\omega = 0.1$, F only starts to

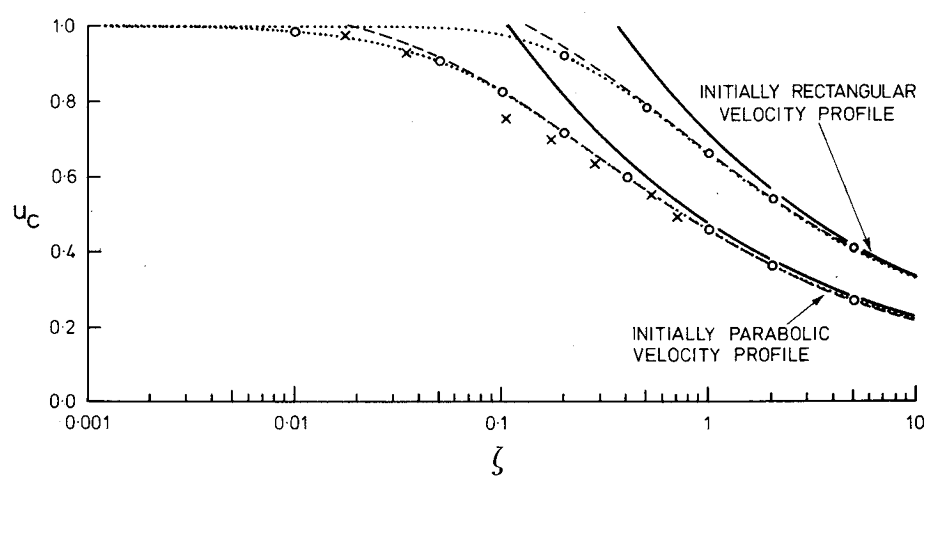

Fig. 1 Variation of nondimensional centerline velocity u_c for steady jet with streamwise distance ζ from the nozzle. ○, Pai and Hsieh; ×, Chanaud & Powell; ...., this investigation; ——, similarity solution with virtual origin at nozzle exit; — — —, similarity solution with virtual origin at $\zeta_v = -0.088$ for initially parabolic velocity profile and $\zeta_v = -0.222$ for initially rectangular velocity profile.

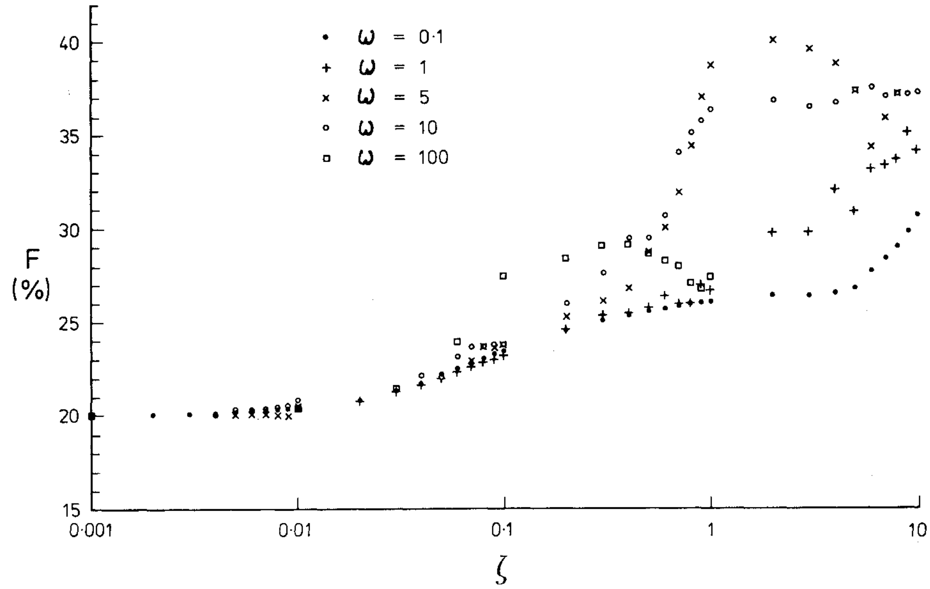

Fig. 2 Variation of percentage peak-to-peak oscillation of centerline velocity (F) with streamwise distance for $\epsilon = 0.1$.

increase sharply for $\zeta > 1$. The accuracy of the numerical technique was extensively evaluated and the authors believe that the behavior of F for $\zeta > 1$ is a correct prediction within the framework of the thin shear layer equations.

Conclusions

The transformation used in solving the thin shear layer equations for the unsteady laminar free jet is successful in reducing substantially the number of grid points in the transverse direction when compared with methods of other workers. It therefore has potential in turbulent jet calculations. Although the unsteady effects, which are important in the prediction of instantaneous quantities, do not influence the mean quantities within the computed frequency range, both theoretical and experimental studies of their influence on the mean flow characteristics at frequencies several orders of magnitude higher than those presented here are warranted.

Acknowledgment

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References

[1] Pai, S. I., *Fluid Dynamics of Jets,* D. Van Nostrand Co., New York, 1954, pp. 76-78.

[2] Schlichting, H., *Boundary-Layer Theory,* 6th Edition, McGraw-Hill, New York, 1966, pp. 170-174.

[3] Pai, S. I. and Hsieh, T., "Numerical Solution of Laminar Jet Mixing With and Without Free Stream," *Applied Scientific Research,* Vol. 27, 1972, pp. 39-62.

[4] Hornbeck, R. W., "Numerical Marching Techniques for Fluid Flows With Heat Transfer," NASA SP-297, 1973, pp. 57-64.

[5] McCormack, P. D., Cochran, D., and Crane, L., "Periodic Vorticity and its Effect on Jet Mixing," *Physics of Fluids,* Vol. 9, 1966, pp. 1555-1560.

[6] Lin, C. C., "Motion in Boundary Layer with a Rapidly Oscillating External Flow," *Proceedings of 9th International Congress of Applied Mechanics,* Vol. 4, 1956, pp. 115-169.

[7] Pai, S. I., "Unsteady Three-Dimensional Laminar Jet Mixing of a Compressible Fluid," *AIAA Journal,* Vol. 3, 1965, pp. 617-621.

[8] Kent, J. C., "Unsteady Viscous Jet Flow into Stationary Surroundings," *Computers & Fluids,* Vol. 1, 1973, pp. 101-117.

[9] Bickley, W., "The Plane Jet," *Philosophical Magazine,* Ser. 7, Vol. 23, 1939, pp. 727-731.

[10] Keller, H. B., "A New Difference Scheme for Parabolic Problems," *Numerical Solutions of Partial Differential Equations,* edited by B. Hubbard, Vol. II, Academic Press, New York, 1979, pp. 327-350.

[11] Cebeci, T. and Bradshaw, P., *Momentum Transfer in Boundary Layers,* McGraw-Hill, New York, 1977, pp. 213-234.

[12] Cebeci, T., "Calculation of Unsteady Two-Dimensional Laminar and Turbulent Boundary Layers with Fluctuations in External Velocity," *Proceedings of the Royal Society of London,* Series A, Vol. 355, 1977, pp. 225-238.

[13] Chanaud, R. C. and Powell, A., "Experiments Concerning the Sound Sensitive Jet," *Journal of the Acoustical Society of America,* Vol. 34, 1962, pp. 907-915.

[14] Lai, J. C. S. and Simmons, J. M., "Numerical Solution of a Periodically Pulsed Two-Dimensional Laminar Free Jet," Res. Rept. 8/78, Dec. 1979, Dept. of Mechanical Engineering, Univ. of Queensland, Brisbane, Queensland, Australia.

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Triple-Point Trajectory of a Strong Spherical Shock Wave

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Introduction

When a spherical shock encounters a planar or conical wall, a shock transition from regular to Mach reflection takes place depending on its angle of incidence, as shown in Fig. 1. This Note presents an approximate method to predict the triple point trajectory of a strong spherical shock over a planar or conical wall.

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